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Estimations in Partial Differential Equations

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ABSTRACT

A general principle for error estimation is described which can be applied to different types of partial differential equations. Particular attention is paid to nonlinear problems. With a programmed procedure based on this estimation principle, error bounds are calculated for boundary value problems involving the differential equation $-\Delta u + f(x,y,u) = 0$.

I cordially thank Dr. James F. Price and Thomas A. Bray. The approximations of the difference method for the examples in Section 8 have been calculated with a program of Dr. Price. T. Bray programmed the general approximation and estimation procedure of Section 7 as well as its application to the examples of Section 8.

This paper will be published in a slightly condensed version in the Proceedings of the IFIP-Congress 65, [14].

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1. The Problem:

This paper is concerned with a certain type of differential inequalities which can be used for different purposes. Here we are mainly interested in the application of such inequalities for error estimations in partial differential equations.

For illustration, consider the boundary value problem

$$\left.\begin{aligned} -\Delta u + f(x^1,x^2,u) &= 0 && \text{on } G, \\ u &= s(x^1,x^2) && \text{on } \Gamma, \end{aligned}\right\} \tag{1.1}$$

where G is a bounded open domain of the (x^1,x^2)-plane and Γ denotes its boundary. Under appropriate assumptions on f and the functions $u(x^1,x^2)$, $v(x^1,x^2)$, it can be shown that the inequalities

$$\left.\begin{aligned} -\Delta u + f(x^1,x^2,u) &\leq -\Delta v + f(x^1,x^2,v) && \text{on } G, \\ u(x^1,x^2) &\leq v(x^1,x^2) && \text{on } \Gamma, \end{aligned}\right\} \tag{1.2}$$

imply $\quad u(x^1,x^2) \leq v(x^1,x^2) \quad$ on $\bar{G} = G \cup \Gamma$.

We are concerned with such implications for a more general problem. We consider a problem with n unknown functions $u^i(x^1,\ldots,x^m)$ $(i = 1,2,\ldots,n)$ of m independent variables. These functions shall be defined and continuous on the closure $\bar{G}$ of a bounded open domain G of the m-dimensional Euclidean space with boundary Γ. With the notations $x = (x^1,\dots,x^m) = (x^i)$, $u = (u^1,\dots,u^n) = (u^k) = u(x)$ the problem shall consist of p equations

$$F^j[u](x) = r^j(x) \qquad \text{on } B_j \qquad (j = 1,2,\dots,p) . \tag{1.3}$$

Each of these equations is a differential equation of at most second order, given on some subset $B_j \subset \bar{G}$. Besides continuity, the functions $u^k(x)$ shall have appropriate differentiability properties depending on the special problem. Let R denote the linear set of all $u(x)$ which have these properties.

Then, under what conditions is the following implication true:

$$\begin{array}{llll} & F^j[u](x) \le F^j[v](x) & \text{on } B_j & (j = 1,2,\dots,p) \\ \text{imply} & u^k(x) \le v^k(x) & \text{on } \bar{G} & (k = 1,2,\dots,n) \end{array} \tag{1.4}$$

for $u, v \in R$.

<u>Example 1</u>: The problem (1.1) can be written in the form (1.3) using:

$$\begin{array}{lll} F^1[u](x) = -\Delta u + f(x^1, x^2, u), & r^1(x) \equiv 0, & B_1 = G; \\ F^2[u](x) = u(x), & r^2(x) = s(x^1, x^1), & B_2 = \Gamma . \end{array}$$

$m = 2$, $n = 1$, $p = 2$. As R, one may choose the set of all functions $u(x)$ which are continuous on $\bar{G}$ and have continuous first and second derivatives on G.

Often, it is of advantage to describe the problem in a shorter abstract form. Let Mu denote the vector

$$Mu = (F^1[u], \ldots, F^p[u]) = (F^j[u])$$

which is an element of the set S of vectors $U = (U^1(x), \ldots, U^p(x))$ with components $U^j(x)$ defined on B_j.

Then, with $r = (r^j)$, the problem (1.3) is

$$Mu = r \ ;$$

and the implication (1.4) can be written as

$$Mu \leq Mv \quad \text{implies} \quad u \leq v, \tag{1.5}$$

where inequalities between vectors of functions are defined in a natural way, namely, as holding component and pointwise.

A problem (1.3) satisfying (1.4) is often called of monotonic type [3], [7], or inverse-monotonic [13]. An operator M with property (1.5) is also said to be inverse-monotonic (more precisely: inverse-isotonic). The reason for this notation is that (1.5) is equivalent to the following statement. The inverse operator M^{-1} exists and is monotonic (isotonic):

$$U \leq V \quad \text{implies} \quad M^{-1}U \leq M^{-1}V,$$

if U, V are in the range of M. Therefore, an equation $Mu = r$ with inverse-monotonic operator M has at most one solution.

We will use the following notations: Derivatives with respect to a variable x^i are denoted by subscripts, for example, $u^j_{ik} = \partial^2 u^j / \partial x^i \partial x^k$. If $n = 1$, we will write $u = u^1$. In this case, a differential operator F^j of the type we consider is

$$F^j[u] = F^j[u](x) = F^j(x,u,u_i,u_{k\ell}) \quad (i,k,\ell = 1,2,\dots,m). \tag{1.6}$$

This equation is to be understood in an obvious way: $F^j[u]$ may depend on all first and second derivatives. In (1.6), the values $u, u_i, u_{k\ell}$ may also be considered as independent variables of a function F^j. For simplicity, it will always be assumed that this function F^j is defined for $x \in B_j$ and all values of the other variables. Similar assumptions shall be made for the other examples of this paper.

2. Application to Error Estimation

Suppose the given problem $Mu = r$ has a solution $u^* \in R$ and the implication (1.5) is true for $u, v \in R$. Then, if $v, w \in R$ satisfy

$$Mw \le r \le Mv,$$

one can conclude that

$$w \le u^* \le v.$$

Given an approximate solution φ with defect $d[\varphi] = -M\varphi + r$, one can try to get such elements v, w in the form:
$v = \varphi - \beta z$, $w = \varphi + \beta z$, where $z \in R$ is suitably chosen and β denotes a real number.

For such v, w, the statement above is equivalent to:

$$M(\varphi - \beta z) - M\varphi \le d[\varphi] \le M(\varphi + \beta z) - M\varphi \tag{2.1}$$

implies

$$-\beta z \le u^* - \varphi \le \beta z.$$

More explicitly, <u>the inequalities</u>

$$F^j[\varphi - \beta z] - F^j[\varphi] \le -F^j[\varphi] + r^j \le F^j[\varphi + \beta z] - F^j[\varphi] \quad \text{on } B_j \quad (j=1,2,\ldots,p) \tag{2.2}$$

<u>imply that</u>

$$|u^{*k} - \varphi^k| \le \beta z^k \qquad \text{on } \bar{G} \quad (k=1,2,\ldots,m) \tag{2.3}$$

<u>Example 1</u>: For the example of Section 1, this implication takes the following form:

$$\begin{aligned} &\beta\Delta z + f(x,\varphi - \beta z) - f(x,\varphi) \le \Delta\varphi - f(x,\varphi) \le -\beta\Delta z + f(x,\varphi + \beta z) - f(x,\varphi) && \text{on } G, \\ &|\varphi - s(x^1,x^2)| \le \beta z(x^1,x^2) && \text{on } \Gamma \end{aligned} \tag{2.4}$$

imply

$$|u^*(x^1,x^2) - \varphi(x^1,x^2)| \le \beta z(x^1,x^2) \qquad \text{on } \bar{G} \tag{2.5}$$

Error estimations of the form (2.2), (2.3) are what we are interested in here. It depends on the type of the problem how φ and the error bounds can be calculated. Usually, the most difficult part is to calculate an approximate solution φ with small defect. We realize that in practical applications, it might not be the solution u^* of the differential equations which is of main interest, but other things, as derivatives, certain linear functionals, etc. In many cases, however, the solution is of interest; and in even other cases, bounds for the solution may help to get other information which one wants to have.

3. Several Proofs

By different methods, several types of problems have been proved to be inverse-monotonic. We sketch some of the typical proofs without stating all necessary assumptions concerning differentiability, etc. The proofs can be applied to more general problems. Some of the results which we will mention do not exactly have the form (1.4) but involve the $<$ sign, also.

Elliptic boundary value problems: We will prove (1.2) under different assumptions.

1) The implication (1.2) is true if on G:

$$f(x,u) < f(x,v) \quad \text{for} \quad u < v . \tag{3.1}$$

Proof: Suppose $w = v - u$ has a negative minimum at $x = \xi \in G$. Then $\Delta w \geq 0$, and consequently, $-\Delta u + f(x,u) > -\Delta v + f(x,v)$ at $x = \xi$ in contradiction to the assumption in (1.2).

In the same way, one can prove that

$$\left.\begin{array}{ll} -\Delta u + f(x,u) < -\Delta v + f(x,v) & \text{on } G \\ u \leq v & \text{on } \Gamma \end{array}\right\} \text{ imply } u \leq v \text{ on } \bar{G},$$

if on G: $f(x,u) \leq f(x,v)$ for $u \leq v$.

Using the strong maximum principle one can show that

2) The implication (1.2) is true if on G:

$$f(x,u) \leq f(x,v) \quad \text{for} \quad u \leq v .$$

Proof [3], [7]: Let w and ξ be as before, and let $K \subset G$ be any neighborhood of ξ such that $w(x) \leq 0$ in K. Then, w accepts its minimum at an inner point of K and satisfies $\Delta w \geq 0$ on K. Therefore, according to the strong maximum principle for elliptic equations, $w(x)$ is constant. This is not possible because $w(\xi) < 0$, and $w \geq 0$ on Γ.

In another proof [2], a contradiction is derived by multiplying the first inequality in (1.2) by w, then integrating over a small enough sphere with center ξ and applying Green's formula.

The statement (1.2) can be proved under even weaker assumptions. For example, in case of a linear operator with $f(x,u) = q(x)u - g(x)$ variational methods have been used [1], [6]. The variational problem which has $-\Delta w + q(x)w = r(x)$ as its Euler equation is shown to possess a unique solution if on $\bar{G}$

$$q(x) \geq -\lambda_1 + \delta,$$

where $\delta = \text{const} > 0$ and λ_1 denotes the smallest eigenvalue of the Dirichlet eigenvalue problem $-\Delta u = \lambda u$. Then, it is proved that in case $r \geq 0$ a function w which has negative values does not solve the variational problem.

The last result can be applied to the nonlinear case by using the mean value theorem of differential calculus, assuming that $\partial f/\partial u$ exists.

3) <u>The implication</u> (1.2) <u>is true if on</u> $\bar{G}$:

$$\frac{\partial f}{\partial u}(x,u) \geq -\lambda_1 + \delta. \tag{3.2}$$

Another way of extending the simple result 1 is by applying the idea of the first proof to $w = (v - u)/z$ instead of $w = v - u$, where $z(x) > 0$ on $\overline{G}$. In case of a differentiable function f, one then comes out with the condition

$$-\Delta z + f_u(x,u)\,z > 0 \qquad \text{on } G$$

instead of (3.1). Assuming appropriate knowledge of the eigenfunction corresponding to λ_1, and choosing z close to such an eigenfunction, one can get the condition (3.2) in this way, also.

Parabolic problems: In order to explain a typical proof for initial value problems, we consider a problem for one dependent variable u ([3][4][8][9][10][12][16]):

4) *The inequalities*

$$u_2 - f(x^1,x^2,u,u_1,u_{11}) < v_2 - f(x^1,x^1,v,v_1,v_{11}) \tag{3.3}$$

$$\text{for } \; 0 < x^1 < 1, \quad 0 < x^2 \le X;$$

$$u < v \quad \text{for} \begin{cases} x^1 = 0,1; & 0 \le x^2 \le X, \\ 0 \le x^1 \le 1; & x^2 = 0 \end{cases} \tag{3.4}$$

together imply

$$u < v \;\; \text{for} \;\; 0 \le x^1 \le 1, \quad 0 \le x^2 \le X,$$

if the function f is isotonic (increasing) with respect to its last variable.

Proof: Suppose that $w = v - u > 0$ for $x^2 < \xi^2$, but $w(\xi^1,\xi^2) = 0$

at a fixed point $\xi = (\xi^1, \xi^2)$. Then, $w_2 \leq 0$, $w_1 = 0$, $w_{11} \geq 0$ at $x = \xi$. These inequalities contradict (3.3) because f is isotonic in u_{11}.

Hyperbolic Equations: In hyperbolic equations, comparably few results are known. In some papers [4], [5], [11], [15], identities are used which can be derived by Green's formula, such as the very simple one (D'Alembert's formula):

$$\iint_{B_x} u_{12}\,dy^1 dy^2 = -g[u] + u(x) \tag{3.5}$$

where

$$g[u] = \tfrac{1}{2}[u(x^1, 1-x^1) + u(1-x^2, x^2)] + \tfrac{1}{2}\int_{1-x}^{y} [u_1 + u_2](y^1, 1-y^1)\,dy^1$$

and B_x consists of all $y = (y^1, y^2)$ such that $1 - y^2 < y^1 \leq x^1$, $1 - x^1 < y^2 \leq x^2$.

5) The inequalities

$$u_{12} + f(x^1, x^2, u) \leq v_{12} + f(x^1, x^2, v) \tag{3.6}$$

$$\text{for} \quad 1 - x^2 < x^1 \leq 1, \quad 0 < x^2 \leq 1,$$

$$\left.\begin{array}{l} u < v \\ u_1 + u_2 < v_1 + v_2 \end{array}\right\} \quad \text{for} \quad x^2 = 1 - x^1,\ 0 \leq x^1 \leq 1 \tag{3.7}$$

together imply

$$u < v \quad \text{for} \quad 1 - x^2 \leq x^1 \leq 1, \quad 0 \leq x^2 \leq 1,$$

if the function f is antitonic with respect to its last variable.

Proof: The inequalities (3.7) imply that $g[u] < g[v]$. Therefore, by integration of (3.6) and using the identity (3.5), it follows that

$$u + \iint_{B_x} f(y^1, y^2, u)\, dy^1 dy^2 < v + \iint_{B_x} f(y^1, y^2, v)\, dy^1 dy^2 \; . \tag{3.8}$$

Suppose now, that $w = v - u > 0$ in $\bar{B}_\xi$, except $x = \xi$ where $w(\xi) = 0$. Then, $w_1 \leq 0$ and $w_2 \leq 0$ at $x = \xi$. These inequalities contradict (3.8) because f is antitonic in u.

Discussion: While some proofs use other means, in most of the proofs given above, contradictions are derived by using conditions on a function w and its derivatives at a certain point ξ. The procedures differ for the different types of problems. For the boundary value problem (see 1 and 2) $w(\xi)$ is minimal, while for the initial value problems, the point ξ --in a certain sense--is the "first" point where $w(\xi) = 0$ (see 4 and 5). Because of this relation $w(\xi) = 0$, for the statement 4 no restrictions are required concerning the dependence of the operator on u, such as, for example, (3.1). (Of course, this proof-technical difference has some deeper reason, which becomes apparent by the formula (3.2) involving an eigenvalue.) The proofs of 4 and 5 require some strong inequalities as (3.4), (3.7). Such implications, involving the $<$-sign, usually can be proved easier and under weaker assumptions; and, in fact, they do not have the same consequences. For example, the existence of M^{-1}, i.e. the uniqueness of a solution does not follow.

The following section contains a unifying approach. For all types of problems considered here, a weaker type of implication involving the $<$-sign can be proved in the same way by using conditions on w at a point ζ where w is minimal and $w = 0$ (Assumption I). The general type of the problem, or the properties of a special problem are then taken into account by the construction of a certain element z (Assumption II). Roughly spoken, in this way the assumptions are split into "local" and "global" conditions.

4. A Unifying Approach

Many of the known results about different types of problems, and new results also, can be gained in a unique way by applying an abstract theorem on inverse-monotonic operators:

Theorem [13]. *Suppose that the following assumptions are satisfied*

I. *For abritrary* $u, \bar{u} \in R$,

$$\left.\begin{array}{l} u \leq \bar{u} \\ Mu \prec M\bar{u} \end{array}\right\} \quad \text{imply} \quad u \prec \bar{u}. \tag{4.1}$$

II. *For a given* $v \in R$ *there exists* $z \in R$ *such that*

$$z \geq 0, \quad Mv \prec M(v + \lambda z) \quad \text{for} \quad \lambda > 0. \tag{4.2}$$

Then, for all $u \in R$,

$$Mu \leq Mv \quad \text{implies} \quad u \leq v. \tag{4.3}$$

Remark: The theorem remains true if all inequalities, except $\lambda > 0$, are reversed.

The notations which occur in the theorem can be defined in an abstract way. For our problem, we define an inequality $u \prec v$ between vectors of functions as component- and point-wise strict inequalities:

$$u \prec v \quad \text{iff} \quad u^k(x) < v^k(x) \quad \text{for all } k \text{ and } x,$$

for which these inequalities have a meaning. Similarly, $Mu \prec Mv$ is defined. Then the assumptions take the following form:

<u>Assumption I:</u> For $u, \bar{u} \in R$, the inequalities

$$u^k(x) \leq \bar{u}^k(x) \qquad (x \in \bar{G};\ k = 1,2,\ldots,n), \tag{4.4}$$

$$F^j[u](x) < F^j[\bar{u}](x) \qquad (x \in B_j;\ j = 1,2,\ldots,p) \tag{4.5}$$

<u>together imply</u>

$$u^k(x) < \bar{u}^k(x) \qquad (x \in \bar{G};\ k = 1,2,\ldots,n)\,. \tag{4.6}$$

<u>Assumption II: There exists</u> $z \in R$ <u>such that</u>

$$z(x) \geq 0 \ \underline{\text{on}}\ G,\ F^j[v](x) < F^j[v + \lambda z](x) \ (x \in B_j,\ j=1,2,\ldots,p;\ \lambda > 0) \tag{4.7}$$

The element z occurring in the error estimation (Section 2) can often be used in Assumption II, and vice versa.

We will give an idea of the proof for the simple special case where (1.3) consists of $p = 2$ ordinary equations for $n = 2$ unknown numbers u^1, u^2. (One may consider these unknowns as constant functions defined on some domain $\bar{G}$). Then, if a vector v is fixed, the vectors $u \leq v$ $(u \prec v)$ constitute a closed (respectively open) quadrant.

First, notice that $z \succ 0$ because of (4.2), (4.1). Let now $Mu \leq Mv$, but not $u \leq v$. Then, the position of the "points" u, Mu, $v + \lambda z$, may be as in Figure 1. The existence of a point $\bar{u} = v + \lambda_0 z$ such as in this figure contradicts Assumption I.

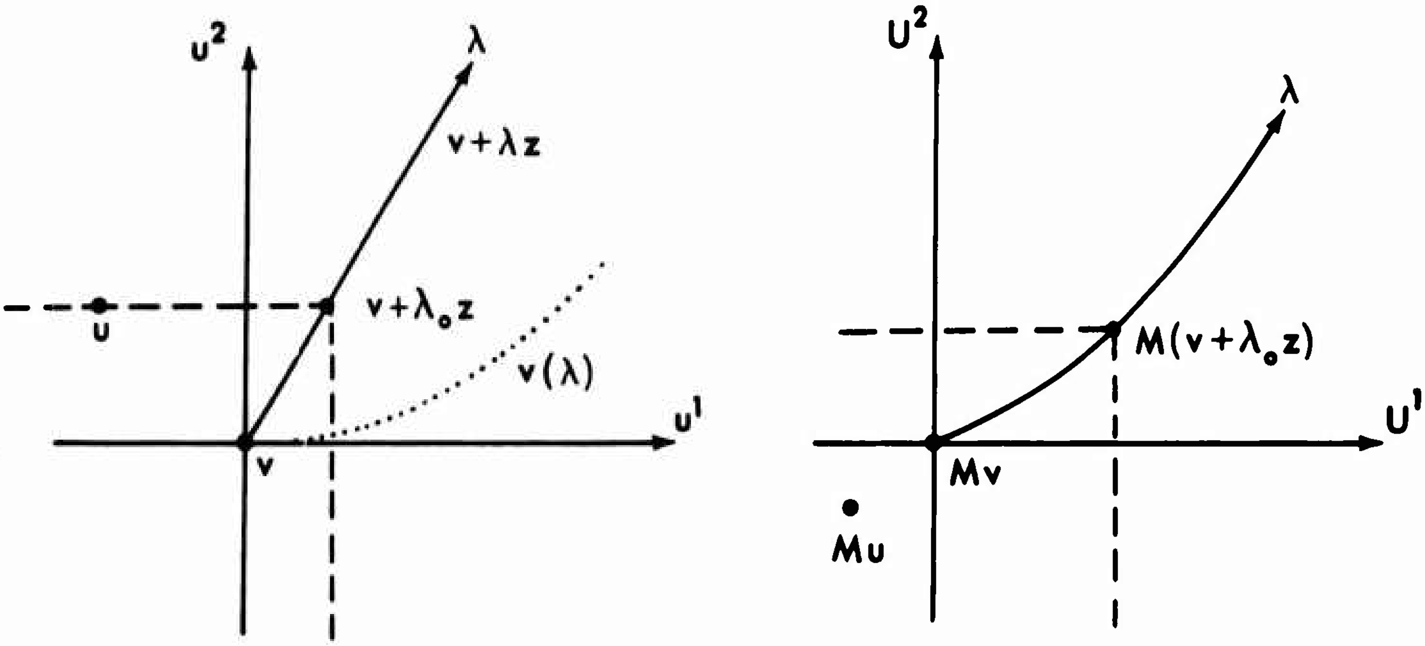

FIGURE 1

This proof shows that the theorem can be generalized in different ways. For example, (4.1) need not be required for all $u, \bar{u} \in R$. Moreover, one may consider a more general curve $v(\lambda)$ instead of $v + \lambda z$ (see Figure 1).

5. Local Assumptions

The general idea of proving Assumption I is as follows. Suppose (4.6) is not true. Then, for some $k = k_0$, $w^k = \bar{u}^k - u^k$ has a minimum $w^k(\xi) = 0$ at some $\xi \in \bar{G}$. This fact yields some conditions on certain first and second order derivatives of w^{k_0} at $x = \xi$. The set of these conditions, together with the inequalities $w^k(x) \geq 0$ (4.4), are then used to derive a contradiction to one of the inequalities in (4.5). For this, there must exist some j_0 such that $F^{j_0}[u]$ contains derivatives of the component u^{k_0}, only. Moreover, the function F^{j_0} must satisfy some "monotonicity-condition" (M-condition).

We give some examples where this proof can be carried through if the given M-conditions are satisfied. It is assumed that the occurring functions possess appropriate differentiability properties. We will not specify the set R of those functions in each case. The examples can be generalized, in particular with respect to the boundary conditions.

Example 2: Generalization of Example 1, boundary value problem for one unknown function u:

$$F^1[u](x) = r^1(x) \quad \text{for} \quad x \in G,$$
$$F^2[u](x) = r^2(x) \quad \text{for} \quad x \in \Gamma$$

with

$$F^1[u](x) = F^1(x, u, u_i, u_{k\ell}) \qquad (i,k,\ell = 1,2,\ldots,m) \quad (5.1)$$
$$F^2[u](x) = u. \qquad (5.2)$$

M-condition:

$$F^1(x,u,u_i,u_{k\ell}) \geq F^1(x,u,u_i,u_{k\ell} + w_{k\ell}) \qquad (5.3)$$

for $x \in G$, all $u, u_i, u_{k\ell}$ and any $m \times m$-matrix

$$(w_{k\ell}) \geq 0,$$

i.e. for any symmetric positive semi-definite matrix $(w_{k\ell})$.

Example 2a: An important special case of (5.1) is the quasi-linear operator

$$F^1[u] = -\sum_{k,\ell=1}^{m} a_{k\ell}(x,u,u_i)u_{k\ell} + c(x,u,u_i)$$

with a coefficient matrix satisfying the M-condition:

$$(a_{k\ell}(x,u,u_i)) \geq 0.$$

In $x \in G$, and all u, u_i.

For example, this condition is satisfied for the equation of minimal surfaces:

$$F^1[u] = -[1+(u_1)^2]u_{11} + 2u_1u_2u_{12} - [1+(u_2)^2]u_{22} = 0,$$

and, if $(u_1)^2 + (u_2)^2 < c^2$, for the equation

$$F^1[u] = -[c^2-(u_1)^2]u_{11} + 2u_1u_2u_{12} - [c^2-(u_2)^2]u_{22} = 0,$$

describing the two-dimensional steady, irrotational flow of a compressible fluid.

Example 3: Initial-boundary value problem for one unknown function:

$$F^1[u](x) = r^1(x) \qquad \text{for } x \in G \cup \Gamma_2, \tag{5.4}$$
$$F^2[u](x) = r^2(x) \qquad \text{for } x \in \Gamma_1,$$

with F^2 as in (5.2) and F^1 a special case of (5.1):

$$F^1[u] = u_m - f(x,u,u_i,u_{k\ell}) \qquad (i,k,\ell = 1,2,\ldots,m-1). \tag{5.5}$$

G shall be a cylinder of points x, such that $(x^1,\ldots,x^{m-1})$ is element of an open bounded domain $\tilde{G}$, and $0 < x^m < X$; Γ_2 is the set of boundary points with $(x^1,\ldots,x^{m-1}) \in \tilde{G}$, $x^m = X$, and $\Gamma_1 = \Gamma - \Gamma_2$.

M-Condition: Condition (5.3) applied to (5.5).

For example, the equation of heat conduction

$$au_4 - (ku_1)_1 - (ku_2)_2 - (ku_3)_3 + f = 0$$

with positive thermal properties $a(x,u)$, $k(x,u)$, and heat generation $f(x,u)$ can be written as (5.4) such that the M-condition is satisfied.

Example 4: Semilinear hyperbolic systems of first order in normal form:

$$F^j[u](x) = -\sum_{k=1}^{n} \alpha^{ji}(x)u_i^j + f^j(x,u) = 0 \quad \text{for} \quad x \in G \cup \Gamma_j, \tag{5.6}$$

$$F^{n+j}[u](x) = u^j(x) = s^j(x) \quad \text{for} \quad x \in \Gamma_{n+j} = \Gamma - \Gamma_j$$

where Γ_j denotes some part of Γ.

M-Condition:

i) $f^j(x,u^1,\ldots,u^n) \geq f^j(x,v^1,\ldots,v^n)$

$$\text{for} \begin{cases} u^k \leq v^k & \text{with} \quad k \neq j \\ u^j = v^j & \end{cases} \qquad (x \in G \cup \Gamma_j;\ j = 1,2,\ldots,n)$$

ii) For each $x \in \Gamma_j$, the "directional derivative"

$$\frac{\partial u^j}{\partial \lambda^j} = \sum_{i=1}^{n} \alpha^{ji}(x)u_i^j,$$

is a derivative into the closed domain $\bar{G}$.

Example 4a (Wave Operator): The differential equation

$$v_{12} + f(x^1, x^2, v, v_1, v_2) = 0$$

can be transformed into a system (5.6)

$$F^1[u] = u^1_1 + u^1_2 - u^2 - u^3 = 0,$$

$$F^2[u] = u^2_2 + f(x^1, x^2, u^1, u^2, u^3) = 0,$$

$$F^3[u] = u^3_1 + f(x^1, x^2, u^1, u^2, u^3) = 0.$$

The first part of the M-condition is satisfied if f is antitonic (decreasing) with respect to the last three variables. For the usual boundary-initial value problems, the second part is satisfied also. A simple example is the characteristic initial value problem for the domain $0 \leq x^1, x^2 \leq 1$ with u^1 given for $x^1 = 0$ and for $x^2 = 0(\Gamma_4)$, u^2 given for $x^1 = 0(\Gamma_5)$, u^3 given for $x^2 = 0(\Gamma_6)$. (The result 5 in Section 3 can be obtained by applying the abstract theorem to the integral operator occurring in (3.8)).

<u>Other Examples</u>: The examples given above can be generalized. Moreover, it is often possible to transform a given problem such that Assumption I becomes satisfied. We describe some of the involved ideas using examples.

1) Consider a problem (1.3) where, for $x \in G$, the equations

$$\begin{aligned} F^1[u] &= (u^1 + u^2)u^1_1 + u^1_2 - u^2 = r^1(x) \\ F^2[u] &= u^2_2 - u^1 = r^2(x), \end{aligned} \qquad (5.7)$$

are given. These equations do not have the form necessary to derive a contradiction to $w^1(\xi) = 0$ at a point $\xi \in G$, because F^1 is not antitonic with respect to u^2. Therefore, introduce new variables U^1, U^2 and replace (5.7) by the system

$$F^{11}[u,U] = u^1u_1^1 - U^2(u_1^1)^+ - u^2(u_1^1)^- + u_2^1 - u^2 = r^1(x),$$

$$F^{12}[u,U] = -F^{11}[-U,-u] = -r^1(x),$$

$$F^{21}[u,U] = F^2[u] = r^2(x), \quad F^{22}[u,U] = -F^2[-U] = -r^2(x),$$

where $2f^+ = |f| + f$, $2f^- = |f| - f$. For $x \in G$, this system has all necessary properties, and, for $U^i = -u^i$, it is equivalent to (5.7). As approximations, respectively bounds, for the new variables U^k, one can use $-\varphi^k$, respectively $-\varphi^k \pm \beta z^k$. Then, the corresponding inequalities (2.2) consist of equivalent pairs, which do not explicitly involve the positive part f^+, or negative part f^- of any function, but $|f|$ instead,

2) If a system (1.3) does not have "normal form", i.e. if not all equations contain derivatives of one variable only, one may get this normal form by transformation of the dependent variables.

Consider a quasi-linear system in matrix notation: $A(x,u)u_1 + u_2 = 0$ on G, together with appropriate boundary conditions. Suppose there exists a nonsingular matrix $\Phi(x,u)$ and a nonsingular diagonal matrix $D(x,u)$ such that $\Phi^{-1}A\Phi = D$. Then, the given system is equivalent to $D\Phi^{-1}u_1 + \Phi^{-1}u_2 = 0$. One can introduce new dependent variables v^k such that $\Phi^{-1}u_1 = \psi v_1 + c$, $\Phi^{-1}u_2 = \psi v_2 + d$, if the function $\psi(x,u)$, and the vectors $c(x,u)$, $d(x,u)$ can be determined such that $(v_1)_2 = (v_2)_1$. Then, the transformed system $D(\psi v_1 + c) + (\psi v_2 + d) = 0$ has normal form. For example, use $v = \Phi^{-1}u$ if Φ does not depend on u.

In the case of the equations

$$u^2 u^1_1 + u^1 u^2_1 + u^1_2 = 0,$$

$$c^2 u^1_1 + u^1 u^2 u^2_1 + u^1 u^2_2 = 0 \quad \text{with} \quad c^2 = [c(u^1)]^2$$

which describe the one-dimensional (isentropic) flow of a compressible fluid, a system in normal form can be gained in the indicated way for the variables $v^1 = F(u^1) + u^2$, $v^2 = -F(u^1) + u^2$, where $F(\rho) = \int_{\rho_0}^{\rho} \frac{c(\sigma)}{\sigma} d\sigma$.

6. Constructing z

Assumption II is not very restrictive, in the sense that something "not very much weaker" must be required. In fact, for linear problems, Assumption II is necessary, in general. For, if a linear operator M is inverse-monotonic, and if for some $r \succ 0$ the equation $Mu = r$ has a solution, then this solution $u = z$ satisfies (4.2)

The element z can be constructed for certain large classes of problems.

Example 2: If the function F^1 in (5.1) is strictly isotonic with respect to u, then $z \equiv 1$ is appropriate. More complicated functions z yield weaker restrictions on F^1. For example, the function $z = \int_r^{r_0} t\, e^{-\rho(r_0 - t)} dt$, has been used [13], where r_0 is the radius of an open sphere $K \supset \overline{G}$ and r denotes the Euclidean distance of its center from x.

Examples 3 and 4a: For initial value problems often very simple functions z are satisfactory. In Example 3, Assumption II is satisfied with $z = e^{Mx^m}$, M large enough, if f obeys a certain one-sided Lipschitz-condition. In Example 4a, a similar condition is sufficient if $z^k = e^{M(x^1 + x^2)}$ is used $(k = 1,2,3)$.

In properly exploiting the theorem one can also gain conditions of more theoretical nature, like (3.2). Consider, for example, the problem (1.1) with $f = q(x)u - g(x)$, and assume that Γ and $q(x)$ are sufficiently smooth. Then, for any such f satisfying (3.2), and $g(x) \equiv 1$, $s(x) \equiv 1$, the problem (1.1) has a solution z. For example $z \equiv 1$ for $q \equiv 1$. Thus, for $q \equiv 1$, the Assumption II is satisfied. But it is also satisfied for any other $f = \tilde{q}u$ satisfying (3.2), because $\tilde{q}(x)$ can be connected with $q \equiv 1$ by a curve $q(x,t) = t\tilde{q}(x) + (1-t)$ $(0 \leq t \leq 1)$. The corresponding $z(x,t)$ depend continuously on t, and $z(x,0) > 0$ on $\bar{G}$. If $z(x,1) > 0$ on $\bar{G}$ would not be satisfied, then for some $t \in (0,1]$: $z(x,t) \geq 0$ on $\bar{G}$, but not $z(x,t) > 0$ on $\bar{G}$. This contradicts Assumption I. Obviously, this is a very special case of far more general results.

7. A Program

The preceding sections have shown that, in principle, the method of error estimation of Section 2 can be applied to many types of problems. Of course, for a concrete problem, usually a lot of additional considerations are necessary to make the method work. To investigate the practical application, a program has been written for the problem (1.1). The general idea is as follows. (For convenience, we use partly different notations than before. For example, we write (x,y) instead of (x^1,x^2); and subscripts do no longer denote derivatives.)

<u>Step D</u>: <u>Calculating Approximations by the Difference Method</u>. (This step is not needed for linear problems.) The ordinary difference method is applied to calculate approximate values $\tilde{u}_{ij}$ at the net-points $(x_i,y_j) \in G$ of a rectangular net. If, for example, the condition (3.2) is satisfied and the meshwidth is not too large, the nonlinear difference equations can be solved by the iterative procedure of Picard or Newton. We have restricted ourselves to cases where the Picard procedure converges $(|f_u| < \lambda_1 - \delta)$, and we have solved the nonlinear systems by a combination of Picard's procedure and the point-overrelaxation method.

<u>Step A</u>: <u>Calculating an Approximation</u> $\varphi(x,y)$. A development $\varphi = \varphi_0 + \alpha_1\varphi_1 + \cdots + \alpha_m\varphi_m$ with properly chosen functions $\varphi_i(x,y)$ is set up. For calculating the constants α_k, the defect $d_1[\varphi] = \Delta\varphi - f(x,y,\varphi)$ is replaced in the net-points (x_i,y_j) by the linear approximation

$$\tilde{d}_1[\varphi] = \Delta\varphi - f(x,y,\tilde{u}) - f_u(x,y,\tilde{u})(\varphi - \tilde{u}) \quad \text{with} \quad \tilde{u}(x_i,y_j) = \tilde{u}_{ij} .$$

In order to get "small" defects $d_1[\varphi]$ on G, and $d_2[\varphi] = -\varphi + s$ on Γ, an orthogonalization method is applied:

$$\gamma_k(\tilde{d}_1[\varphi],\psi_k)_1 + \delta_k(d_2[\varphi], \vartheta_k)_2 = 0 \qquad (k=1,2,\ldots,m), \qquad (7.1)$$

where ψ_k and ϑ_k are properly chosen functions, the constants γ_k and δ_k are only introduced for computational convenience, and $(\ ,\ )_1$, $(\ ,\ )_2$ denote discrete inner products. For example, $(u,v)_1 = \Sigma\, w_{ij} u(x_i, y_j) v(x_i, y_j)$ with given weights, involving all net-points in $\bar{G}$. The linear system (1.7) for the α_k is solved by an elimination method.

Step E: Calculating an Error Bound. For a chosen function z, one determines a constant β_0 such that

$$|d_1[\varphi]| \le \beta_0\{-\Delta z + f_u(x, y, \varphi) z\} \text{ on } G, \quad |d_2[\varphi]| \le \beta_0 z \text{ on } \Gamma.$$

Practically, this is done for points (x,y) in a finer net. Then, if β_0 is small enough, the desired inequalities (2.4) hold for a number β, somewhat larger than β_0, say $\beta = 1.01\beta_0$. This has to be checked. If this is so, then the error estimation (2.5) holds.

8. Numerical Examples

The program of Section 7 has been used to calculate approximations and corresponding error bounds for the following problems:

1. $-\Delta u = 1$ for $|x|, |y| < 1$,

   $u = -\frac{1}{2\pi^2}(\cos \pi x + \cos \pi y)$ on the boundary.

2. $-\Delta u = 1$ for $|x|, |y| < 1$,

   $u = 0$ on the boundary.

3. $-\Delta u = e^u$ for $|x|, |y| < 1$,

   $u = 0$ on the boundary.

4. $-\Delta u = -e^u$ for $|x|, |y| < 1$,

   $u = 0$ on the boundary.

The problems 1 and 2 were mainly calculated to check how the method works, before starting the nonlinear problems 3 and 4. The general program is constructed to handle more complicated problems. However, we did not accomplish to compute more examples during the available time. In all of these problems we chose a square net with mesh width $h = 0.04$.

In Problem 1, we used the development

$$\varphi = \varphi_0 + (1 - x^2)(1 - y^2)(\alpha_1\omega_1 + \cdots + \alpha_m\omega_m) \tag{8.1}$$

with polynomials ω_i having appropriate symmetry properties:

$$1,\ x^2 + y^2,\ \ x^2y^2, \ldots, \tag{8.2}$$

and

$$\varphi_0 = -\frac{1}{2\pi^2}(\cos \pi x + \cos \pi y).$$

As functions ψ_k, there were used orthogonal polynomials

$$P_0,\ P_2(x) + P_2(y),\ \ P_2(x)P_2(y), \ldots \tag{8.3}$$

where $P_i(x)$ is proportional to the i^{th} Legendre polynomial. Finally

$$\gamma_k = 1,\quad \delta_k = 0 \quad (k=1,2,\ldots,m);\quad w_{ij} = 1 \quad (i,j=1,2,\ldots,m), \tag{8.4}$$

and in Step E

$$z = 2 - (x^2 + y^2). \tag{8.5}$$

In this way, we got the following error estimations for Problem 1.

For

$m = 4$: $|u^*(x,y) - \varphi(x,y)| \leq 0.0093\ (2 - x^2 - y^2) \quad (|x|, |y| \leq 1)$

where $\varphi(0,0) = 0.326\ 708$;

$m = 6$: $|u^*(x,y) - \varphi(x,y)| \leq 0.0064\ (2 - x^2 - y^2) \quad (|x|, |y| \leq 1)$

where $\varphi(0,0) = 0.326\ 834$;

$m = 9$: $|u^*(x,y) - \varphi(x,y)| \leq 0.000270\ \ (2 - x^2 - y^2) \quad (|x|, |y| \leq 1)$.

In a first run we had tried with the polynomials in (8.2) as functions ψ_k instead of (8.3). But then, the linear systems which determine the constants α_k became ill-conditioned for $m = 6$ and $m = 9$, and the corresponding approximations φ were less accurate than the approximation for $m = 4$.

For Problem 2, we did not apply the approximation and estimation procedure immediately, because Δu is discontinuous at the corners of the domain. The singularities were removed by introducing a new variable

$$v = u + p, \tag{8.6}$$

where $p(x,y)$ satisfies

$\Delta p = 0$ for all x,y, except at the corners,

$\Delta p = 1$ at the corners.

The function $p(x,y)$ consists of four summands of the type $\pi^{-1}\mathrm{Im}(z^2 \log z)$, each of them belonging to one of the corners. For example, $z = 1 + x + i(1 + y)$ for the corner $(x,y) = (-1,-1)$. More explicitly,

$$p(x,y) = \frac{1}{\pi}(p_1 + p_2 + p_3 + p_4)$$

with

$$p_1 = q(x,y), \quad p_2 = q(-y,x), \quad p_3 = q(-x,-y), \quad p_4 = q(y,-x)$$

and

$$q(x,y) = (1 + x)(1 + y) \log [(1 + x)^2 + (1 + y)^2]$$
$$+ [(1 + x)^2 - (1 + y)^2] \operatorname{arctg} \frac{1 + y}{1 + x} .$$

The transformed Problem 2' then is

$$-\Delta v = 1 \quad \text{for} \quad |x|,|y| < 1,$$

$$v = p(x,y) \quad \text{at the boundary.} \tag{8.7}$$

The procedure in Section 7 was applied to this problem. We again used a development (8.1) with ω_i as in (8.2), but

$$\varphi_0 = \frac{1}{\pi}[H(x) + H(y) - H(1)]$$

with

$$H(x) = h(x) + h(-x)$$

and

$$h(x) = 2(1 + x) \log [4 + (1 + x)^2]$$
$$+ [4 - (1 + x)^2] \operatorname{arctg} \tfrac{1}{2}(1 + x) - \pi.$$

This function φ_0 satisfies the boundary condition (8.7).

With functions ψ_k as in (8.3), the quantities in (8.4), and z in (8.5), we got the following error estimation for $m = 4$:

$$|v^*(x,y) - \varphi(x,y)| = |u^*(x,y) + p(x,y) - \varphi(x,y)| \leq$$
$$\leq 0.000\,002 \quad (2 - x^2 - y^2) \quad (|x|,|y| \leq 1)$$

where

$$\varphi(0,0) = 1.177\,227\,9,$$
$$p(0,0) = 0.882\,542\,4,$$
$$\varphi(0,0) - p(0,0) = 0.294\,685\,5.$$

Of course, the error bound does not take into account the rounding errors which occurred during the calculation of $\varphi(x,y)$ according to (8.1).

For the linear Problems 1 and 2, we could also have used functions φ which satisfy the differential equation instead of the boundary condition. (Then, $\gamma_k = 0$, $\delta_k = 1$ $(k=1,2,\ldots,m)$). But, this is not possible for the nonlinear problems 3 and 4, and, as mentioned above, the linear problems mainly served to check the procedure.

The Problem 3 also was transformed using (8.6) before applying the approximation and estimation procedure. The transformed Problem 3' is:

$$-\Delta v = e^{-p(x,y)} e^{v} \quad \text{for} \quad |x|, |y| < 1,$$

$$v = p(x,y) \quad \text{on the boundary.}$$

In Step A and Step E, we used the same quantities as in Problem 2'. In this way, we got the following results for $m = 6$:

$$|v^*(x,y) - \varphi(x,y)| = |u^*(x,y) + p(x,y) - \varphi(x,y)| \leq 0.001\ 02(2 - x^2 - y^2)$$

for $|x|, |y| \leq 1$,

where: $\varphi(0,0) = 1.278\ 072\ 3$,

$\varphi(0,0) - p(0,0) = 0.395\ 529\ 9$,

and the approximations $\tilde{v}(0,0)$ for $v^*(0,0)$, $\tilde{u}(0,0)$ for $u^*(0,0)$, obtained by the difference method, are

$$\tilde{v}(0,0) = 1.277\ 974\ 7,$$

$$\tilde{u}(0,0) = \tilde{v}(0,0) - p(0,0) = 0.395\ 432\ 3.$$

In Problem 4 we proceeded similarly, except that now the transformation

$$v = u - p$$

was used. The results for $m = 6$ were:

$$|v^*(x,y) - \varphi(x,y)| = |u^*(x,y) - p(x,y) - \varphi(x,y)| \leq 0.00038\ (2 - x^2 - y^2)$$

where:

$$\varphi(0,0) = -1.127\ 658\ 9,$$

$$\varphi(0,0) + p(0,0) = -0.245\ 116\ 5,$$

$$\tilde{v}(0,0) = -1.127\ 576\ 2,$$

$$\tilde{u}(0,0) = \tilde{v}(0,0) + p(0,0) = -0.245\ 033\ 8.$$

9. References: (Additional references can be found in the books and papers listed below.)

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